MATRICES

1.- Calculate the following determinants:

a) $\begin{vmatrix} 3 & 2 & 5 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{vmatrix}$ b) $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 5 \\ 6 & 0 & -2 \end{vmatrix}$ c) $\begin{vmatrix} 3 & 1 & 5 & 0 \\ 5 & 4 & 6 & 3 \\ 1 & 3 & 2 & 1 \\ 6 & 7 & 5 & 4 \end{vmatrix}$

d) $\begin{vmatrix} 7 & 6 & 8 & 5 \\ 6 & 7 & 10 & 6 \\ 7 & 8 & 8 & 9 \\ 8 & 7 & 9 & 6 \end{vmatrix}$ e) $\begin{vmatrix} 1 & 3 & 2 & 1 \\ 3 & 5 & 3 & 2 \\ 3 & 6 & 2 & 2 \\ 6 & 4 & 5 & 3 \end{vmatrix}$ f) $\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$

g) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$ h) $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$ i) $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$

j) $\begin{vmatrix} 3 & 4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ k) $\begin{vmatrix} 3 & 4 & 1 & 2 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ l) $\begin{vmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 & 1 \end{vmatrix}$

m) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 3 & 4 & 5 \\ -1 & -2 & 0 & 4 & 5 \\ -1 & -2 & -3 & 0 & 5 \\ -1 & -2 & -3 & -4 & 0 \end{vmatrix}$ n) $\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{vmatrix}$ ñ) $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix}$

o) $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$ p) $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$ q) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$

r) $\begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & 2 \\ 1 & 2 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix}$ s) $\begin{vmatrix} 1 & 1 & 1 & 1 & . & . & 1 \\ -1 & a & 1 & 1 & . & . & 1 \\ -1 & -1 & a & 1 & . & . & 1 \\ . & . & . & . & . & . & . \\ -1 & -1 & -1 & -1 & . & . & a \end{vmatrix}$

t) $\begin{vmatrix} 1 & 2 & 3 & . & . & n-1 \\ 2 & 3 & 4 & . & . & n \\ 3 & 4 & 5 & . & . & n+1 \\ . & . & . & . & . & . \\ n-2 & n-1 & n & . & . & 2n-2 \\ n-1 & n & n+1 & . & . & 2n-1 \end{vmatrix}$

2.- Solve the equation $\begin{vmatrix} 4 & x & 6 \\ 5 & 7 & 12 \\ 3 & -1 & x \end{vmatrix} = 0$.

3.- Show that:

a) $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = -2(x^3+y^3)$ b) $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix} = x^2z^2$

4.- Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 6 & -3 & 2 \\ 4 & 1 & 3 \end{pmatrix}$. Calculate the determinant $|A - \lambda I_3|$, with $\lambda \in \mathbb{R}$.

5.- Knowing that $\begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1$, obtain the value of $\begin{vmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+4 & y+4 & z+4 \end{vmatrix}$.

6.- Let $A, B \in M_4$ with $|A|$=3, $|B|$=-2. Calculate:

a) $|2A|$.

b) $|\frac{1}{2}B|$.

c) $|BA^t|$.

d) $|(BA)^t|$.

e) $|(B^tA^tB)^t|$.

7.- Calculate the rank of the following matrices:

$$A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 3 \\ 4 & 1 & 0 \\ 10 & 1 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & -1 & 0 & 1 & 3 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & -1 & 3 & 6 \\ 5 & -2 & -1 & 4 & 9 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & -4 & 3 & 4 & 1 & 8 \\ 1 & 2 & 0 & -2 & 1 & -1 \\ 3 & 6 & 3 & -6 & 6 & 3 \\ 2 & 4 & 2 & -4 & 4 & 2 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & -2 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 1 \\ 1 & 2 & 0 & -2 & 1 \\ 2 & 1 & 3 & -3 & 4 \\ 3 & -2 & 2 & -2 & 5 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 3 & 3 & 5 \\ 0 & 3 & 4 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & 1 & -2 & 1 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 2 & 1 \\ 3 & 2 & 1 & 2 \end{pmatrix} \qquad H = \begin{pmatrix} 0 & 2 & 1 & -1 & 2 \\ 1 & 0 & -1 & 3 & 2 \\ 0 & 4 & 2 & -2 & 4 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

8.- Calculate the rank of the following matrices depending on the values of the real parameter a:

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 3 & 2 & -1 & 3 \\ a & 3 & -2 & 0 \\ -1 & 0 & -4 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & -1 & 1 \\ a & 1 & 1 & 1 \\ 1 & -1 & 3 & 0 \\ 4 & 2 & 0 & a \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 5 & 1 & a \end{pmatrix}$$

9.- Study if the following matrices are invertible. If affirmative, calculate the inverse matrix.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \\ 4 & 5 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 3 & 1 \\ -3 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 4 & -2 \\ -2 & 6 & -8 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad G = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

10.- Study the existence of the inverse matrix depending on the values of $m \in \mathbb{R}$. Calculate the inverse matrix in the invertible cases.

$$A = \begin{pmatrix} m & -1 & 0 \\ 1 & m-1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} m & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{pmatrix}$$

11 .- Study for what values of the parameter $m \in \mathbb{R}$, the following matrices do not have inverse.

$$A = \begin{pmatrix} 1-m & 2 \\ 3 & 2-m \end{pmatrix} \qquad B = \begin{pmatrix} -m & 5 \\ 2 & 3-m \end{pmatrix} \qquad C = \begin{pmatrix} 2-m & 3 & 1 \\ 1 & 1-m & 4 \\ 0 & 1 & 1-m \end{pmatrix}$$

$$D = \begin{pmatrix} m & 9 & 4 \\ 4 & m & -1 \\ 7 & 7 & 7 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 1 & 1 \\ m & 1 & 0 \\ -1 & 3 & m-1 \end{pmatrix}$$

12.- Given the matrices $P=\begin{pmatrix}1&1&1\\1&0&1\\0&1&1\end{pmatrix}$ and $B=\begin{pmatrix}1&2&-1\\0&-3&1\\2&0&1\end{pmatrix}$, calculate the matrix A that verifies $P^{-1}AP=B$.

13.- Solve, using Cramer's Rule:

a) $\left.\begin{aligned}x+y-z&=3\\5x-y+2z&=5\\-3x+3y-4z&=1\end{aligned}\right|$ b) $\left.\begin{aligned}x+y-z&=1\\x+2y+z&=2\\x+3y-z&=0\end{aligned}\right|$ c) $\left.\begin{aligned}x+y-2z+t+3s&=1\\2x-y+2z+2t+6s&=2\\3x+2y-4z-3t-9s&=3\end{aligned}\right|$

14 .- Discuss the following systems of linear equations as the real parameters (*m, a and b)* varies, and solve them using Cramer's Rule:

a) $\left.\begin{aligned}mx+y-z&=1\\x+2y+z&=2\\x+3y-z&=0\end{aligned}\right|$ b) $\left.\begin{aligned}mx+y+z&=m\\x+my+z&=1\\x+y+mz&=1\end{aligned}\right|$ c) $\left.\begin{aligned}x+y+z&=m\\x+(1+m)y+z&=2m\\x+y+z&=4\end{aligned}\right|$

d) $\left.\begin{aligned}x+my-z&=0\\2x-3y-2z&=0\\x+2y+z&=0\end{aligned}\right|$ e) $\left.\begin{aligned}mx+y+3z&=3\\x-y-z&=0\\5x-3y-2z&=6\end{aligned}\right|$ f) $\left.\begin{aligned}x-2y+z&=-1\\x+y+3z&=4\\5x-y+mz&=10\end{aligned}\right|$

g) $\left.\begin{aligned}-x+2y-2z&=0\\2x-y+az&=b\\2x-2y+3z&=1+b\end{aligned}\right|$ h) $\left.\begin{aligned}2x+ay+z&=7\\x+ay+z+t&=b\\x+2ay+t&=-1\\bx+ay&=b\end{aligned}\right|$

15.- Given the matrix $A=\begin{pmatrix}1&1&m\\3&2&4m\\2&1&3\end{pmatrix}$ with $m\in\mathbb{R}$:

a) For what values of the parameter *m*, is the matrix *A* invertible?

b) Solve the linear system $AX=0_3$ using the results of point a).

16.- Consider the matrices $A=\begin{pmatrix}1&a&0&0\\a&1&0&1-a\\0&0&1&0\end{pmatrix}$, $B=\begin{pmatrix}a\\a^2\\a^3\end{pmatrix}$ and $X=\begin{pmatrix}x\\y\\z\\t\end{pmatrix}$, with *a* being a real parameter.

a) Discuss the system $AX=B$ as *a* varies.

b) In the compatible cases, calculate the solutions of $AX=B$.

17 .- In a market with perfect competition the functions of supply and demand of the goods are given by:

$$Q_{1d} = 10 - 2P_1 + 4P_2 \qquad Q_{2d} = 10 + 5P_1 - 3P_2$$

$$Q_{1s} = 20 + 3P_1 - 2P_2 \qquad Q_{2s} = 30 - 7P_1 + 5P_2$$

Where Q_{id} is the demanded quantity of good i, Q_{is} is the supplied quantity of good i and P_i is the price of the good i, for i=1,2. Calculate the prices for which the market is in equilibrium and the demanded and supplied quantities of each good in this situation.

18 .- The condition of equilibrium for the price of three goods in a given market is determined by the following equations:

$$\begin{aligned} 11P_1 - P_2 - P_3 &= 31 \\ -P_1 + 6P_2 - 2P_3 &= 26 \\ -P_1 - 2P_2 + 7P_3 &= 24 \end{aligned}$$

where P_1, P_2 and P_3 are the prices of these three goods. Calculate the price of equilibrium for each good.

DIAGONALIZATION OF SQUARE MATRICES

1.- Consider the vectors $u = (1, -3, 2)$ and $v = (2, -1, 1)$ of $\mathbb{R}^3$:

a) Write, if it is possible, the vectors (1, 7, -4) and (2, -5, 4) as a linear combination of u and v.

b) For which values of x is the vector $(1, x, 5)$ a linear combination of u and v?

2.- The vectors $v_1 = (1, 1, -1)$, $v_2 = (2,1,3)$ and $v_3 = (5, 2, 10)$ of $\mathbb{R}^3$, are linearly independent? If not, find the relation of dependency.

3.- Given the vectors $u_1 = (2,-1,0)$, $u_2 = (0,1,-1)$ and $u_3 = (8,3,1)$ of $\mathbb{R}^3$, study if they are linearly dependent or independent.

4.- Given the vectors $u_1 = (1,0,-1,0)$, $u_2 = (2,0,3,-1)$, $u_3 = (1,1,-1,1)$ and $u_4 = (2,1,-2,1)$ of $\mathbb{R}^4$, study if they are linearly independent. If not, find the relation of dependency.

5.- Given the vectors of $\mathbb{R}^4$ $v_1 = (1,1,0,m)$, $v_2 = (3,-1,n,-1)$ and $v_3 = (-3,5,m,-4)$, determine the values of the parameters m and n so that the three vectors be linearly dependent.

6.- Sean $u = (-1, 0, 0)$, $v = (1, 1, 0)$ and $w = (-1, 1, -1)$ vectors of $\mathbb{R}^3$:

a) Show that $\{u, v, w\}$ is a basis of $\mathbb{R}^3$.

b) Find the coordinates with respect to this basis of the vector whose coordinates with respect to the canonical basis are 1, 0, 2.

c) Find the coordinates concerning the canonical basis of the vector $a = 3u - v + 5w$

7.- Let $A \in M_n$, $\lambda \in \mathbb{R}$ an eigenvalue of A and $\mathbf{x} \in \mathbb{R}^n$ an eigenvector of A associated to λ

Prove that:

a) $\alpha\lambda$ is an eigenvalue of the matrix αA for any $\alpha \in \mathbb{R}$ and $\mathbf{x}$ is an eigenvector of αA associated to $\alpha\lambda$.

b) λ^p is an eigenvalue of A^p and $\mathbf{x}$ is an eigenvector of A^p associated to λ^p, with $p \in N$.

c) $|A| = 0 \Leftrightarrow \lambda = 0$ is an eigenvalue of A.

d) If A invertible then $\lambda \neq 0$. Moreover, λ^{-1} is an eigenvalue of A^{-1} and $\mathbf{x}$ is an eigenvector of A^{-1} associated to λ^{-1}.

8.- Let $A, B \in M_n$ be similar matrices. Prove that:

a) $|A| = |B|$.

b) A^p is similar to B^p for any $p \in N$.

c) If A is invertible then B is invertible and A^{-1} is similar to B^{-1}.

9.- Let $A \in M_n$. Prove that A and A^t have the same characteristic polynomial.

10.- Given the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$ answer the following questions:

a) Study if 3 is eigenvalue of A or not.

b) Are the vectors (1,1,1) and (0,0,1) eigenvectors of A? If in the affirmative, find the associated eigenvalues.

11.- Given the matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -3 & 1 \end{pmatrix}$ answer the following questions:

a) Study if the vector $(-1, \frac{1}{3}, \frac{1}{2})$ is an eigenvector or not of the matrix A. If in the affirmative, determine the associated eigenvalue.

b) The same question for the vector (-1,0,1).

12.- For each one of the following matrices reason out if it is diagonalizable or not. Besides:

a) If in the affirmative, give a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

b) If in the negative, calculate the eigenvalues and the eigenvectors associated to each eigenvalue.

$$A_1 = \begin{pmatrix} 0 & 3 & 0 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 4 & -4 & 6 \\ 3 & -4 & 6 \\ 1 & -2 & 3 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 2 & 0 & -2 \\ -3 & -1 & 2 \\ 2 & 0 & -2 \end{pmatrix} \qquad A_8 = \begin{pmatrix} 2 & 4 & 3 \\ -2 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} \qquad A_9 = \begin{pmatrix} -1 & 2 & -2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

13.- A matrix $A \in M_2$ verifies the following conditions: $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and (2,-1) is an eigenvector of A associated to the eigenvalue $\lambda = -2$. Find the matrix A indicating if it is diagonalizable or not. If affirmative, give a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

14.- Let $A = \begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix}$ with $a, b \in \mathbb{R}$. Calculate the values of the parameters a and b so that the vector (2,-1) is eigenvector of A associated to the eigenvalue 2.

15.- Find a matrix $A \in M_3$ of eigenvalues -1, 1, 2 with associated eigenvectors (1,0,-1), (1,1,0), (3,-3,1), respectively.

16.- Find a matrix $A \in M_3$ with eigenvalue $\lambda_1 = 1$ (simple) and eigenvalue $\lambda_2 = 2$ (double) with associated eigenvectors (1,1,0), (1,0,0), respectively.

17.- Find a matrix $A \in M_3$ with eigenvalue $\lambda_1 = -1$ (double) and eigenvalue $\lambda_2 = 3$ (simple) with associated eigenvectors (1,0,2), (-1,0,0) and (0,1,1), respectively.

18.- Let $A = \begin{pmatrix} 3 & 2 \\ a & b \end{pmatrix}$ with $a, b \in \mathbb{R}$. Calculate the values of the parameters *a* and *b* so that *A* has eigenvalues 1 and -1. Is *A* a diagonalizable matrix?

19.- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & a \\ 3 & 0 & 1 \end{pmatrix}$ with $a \in \mathbb{R}$.

a) For which values of the parameter *a is* $\lambda = -2$ an eigenvalue of *A*?

b) For which values of the parameter *a* is the matrix *A* diagonalizable?

20.- Determine a matrix $A \in M_3$ such that $A \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ and that his eigenvectors be the non-null vectors of $\mathbb{R}^3$ belonging to the sets $\{(x, y, z) \in \mathbb{R}^3 \mid x = z\}$, $\{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0, z = 0\}$.

21.- The matrix $A = \begin{pmatrix} a & 1 & p \\ b & 2 & q \\ c & -1 & r \end{pmatrix}$ admits as eigenvectors (-1,-1,0), (1,0,-2) and $(0, -1, 1)$ associated to the eigenvalues 3, 0 and 3/2 respectively. Answer the following items:

a) Find the unknown elements of *A*.

b) Is *A* diagonalizable? If affirmative, perform the diagonalization.

22.- Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}$

a) Find its eigenvalues and eigenvectors. It is *A* diagonalizable?

b) Check that that the determinant of the matrix *A* is the product of its eigenvalues.

23.- Check that the matrices $A=\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ and $B=\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{pmatrix}$ have the same eigenvalues but, however, are not similar.

24.- Calculate A^{100} and, in general, A^k, with $k \in N$, for the matrix $A=\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.

25.- Calculate A^k with $k \in N$ odd, for the matrix $A=\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

26.- Calculate A^n $\forall n \in N$ in each one of the following cases:

a) $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ b) $A=\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$

27.- Calculate a symmetric matrix $A \in M_3$ that verify: v=(1,-1,0) is eigenvector of A, $A\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $|A|$ =0. Calculate A^{50}.

28.- Determine for which values of the parameters $b, c \in \mathbb{R}$ the following matrices are diagonalizable. If affirmative, find a diagonal matrix similar to the given one.

$A_1=\begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & b \\ 3 & 0 & c \end{pmatrix}$ $A_2=\begin{pmatrix} b & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $A_3=\begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

29.- For each one of the following matrices A_i, i=1,2,3, find, if it is possible, an invertible matrix P and a diagonal matrix D so that $D=P^{-1}A_iP$.

$A_1=\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$ $A_2=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $A_3=\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

30.- The characteristic polynomial of a matrix A of order 3 is $P(\lambda)=-\lambda^3+21\lambda-20$. With this information, can you justify whether A is invertible and/or diagonalizable?

31.- The eigenvalues of a certain diagonalizable matrix A $\in$M(n $\times$ n) are the roots of the characteristic polynomial $P(\lambda) = \lambda^5 + \lambda^4 - 5\lambda^3 - 5\lambda^2 + 4\lambda + 4$.

a) Determine the eigenvalues and their multiplicity.

b) Determine the dimension of the matrix A and Rk(A) .

c) Calculate, if possible, $\left|A^{-1}\right|$ and $\left|-\frac{1}{2}A^{-1}\right|$.

d) Calculate the eigenvalues of A^2 and their multiplicity.

32.- Consider the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

a) Calculate the eigenvalues and eigenvectors of A.

b) Is it diagonalizable? If it is, calculate a similar diagonal matrix D and a matrix P such that $D = P^{-1}AP$.

c) Is there any value of the parameter *a* so that (3, -6, a) is an eigenvector of A?

33.- Let A be a matrix of order 3 whose characteristic polynomial is

$$P(\lambda) = -\lambda^3 - 2\lambda^2 + 5\lambda + 6 :$$

a) Calculate the eigenvalues of A and the determinant of A.

b) Explain if the following statements are true or false:

i) The homogeneous linear system, AX = 0, is consistent and determined.

ii) There is a diagonal D matrix and a regular P matrix such that $D = P^{-1}AP$.

34.- Given a symmetric matrix A of order 4 whose eigenvalues are 1, 3, -5 and -5:

i) Reason if A^{-1} exists and, if so, calculate its determinant.

ii) Obtain the rank of the matrix $A + 5I_4$?

35.- Let $A=\begin{pmatrix} m & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ with m being a real number:

a) Study the existence of the inverse matrix of A according to the values of the parameter m. Calculate, when possible, A^{-1}.

b) Determine the values of the parameter m so that A is diagonalizable.

c) Calculate the characteristic polynomial and the eigenvalues of A.

d) For m = 0, calculate a diagonal matrix D and a matrix P such that $D=P^{-1}AP$.

36.- Given the matrix $A=\begin{pmatrix} -1 & 0 & -3 \\ 3 & 3 & 3 \\ -3 & 0 & -1 \end{pmatrix}$,

a) Calculate the eigenvalues of A.

b) Calculate the eigenvectors of A.

c) Is it diagonalizable? If it is, calculate a regular matrix P and a diagonal matrix D such that $D=P^{-1}AP$.

d) Calculate the eigenvalues of A^3.

QUADRATIC FORMS

1.- Consider the quadratic form $Q(\mathbf{x})$ of associated matrix (respectively)

$$A_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

Answer to the following points:

a) Express $Q(\mathbf{x})$ in polynomial form.

b) Find, by the method of eigenvalues, a diagonal expression for Q ($\mathbf{x}$).

c) Classify $Q(\mathbf{x})$.

2.- Consider the quadratic form Q($\mathbf{x}$) of associated matrix (respectively)

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Answer to the following points:

a) Express $Q(\mathbf{x})$ in polynomial form.

b) Find, by completing squares, a diagonal expression for Q ($\mathbf{x}$).

c) Classify $Q(\mathbf{x})$.

3.- Consider the quadratic form $Q(x, y, z) = 2x^2 + 5y^2 + 5z^2 + 4xy - 4xz - 8yz$ and answer:

a) Find a diagonal expression for Q.

b) Classify Q.

4.- For each one of the following matrices, answer:

a) Find a congruent diagonal matrix to the matrix given.

b) Classify the quadratic form that it represents.

$$A_1 = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad A_3 = \begin{pmatrix} -2 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & -2 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 5 & -2 & 0 & -1 \\ -2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -1 & 1 & -2 & 2 \end{pmatrix} \qquad A_5 = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}$$

$$A_6=\begin{pmatrix}1&0&1\\0&4&1\\1&1&4\end{pmatrix}\qquad A_7=\begin{pmatrix}2&2&-2\\2&3&-4\\-2&-4&0\end{pmatrix}$$

5.- Classify by their sign the following quadratic forms:

a) $Q(x,y)=3x^2-4xy+7y^2$.

b) $Q(x,y)=x^2+y^2+2xy$.

c) $Q(x,y)=6xy-2x^2-5y^2$.

d) $Q(x,y)=4y^2+8xy$.

e) $Q(x,y,z)=y^2+2xy+2yz$.

f) $Q(x,y,z)=x^2+4y^2+z^2+2xy+2xz+4yz$

g) $Q(x,y,z)=x^2+2y^2+z^2+xz+2xy+2yz$.

h) $Q(x,y,z)=4x^2+4y^2+z^2-4xy$.

i) $Q(x,y,z)=2xy+2xz+2yz$.

j) $Q(x,y,z)=x^2+y^2+z^2$.

k) $Q(x,y,z)=-4x^2+y^2+3z^2+3xz+yz$.

l) $Q(x,y,z)=2x^2+2xz+3y^2+2z^2$.

m) $Q(x,y,z)=2x^2-y^2+3z^2-3xy+4yz-2xz$.

n) $Q(x,y,z)=x^2+10y^2+6xy$.

o) $Q(x,y,z)=x^2+4y^2+3z^2+4xy+2xz+4yz$.

p) $Q(x,y,z,t)=2xz-3yt+2xt-t^2$.

q) $Q(x,y,z)=x^2-y^2-2z^2+2xy+4yz$.

r) $Q(x,y,z)=2xy+4yz-4xz-x^2-y^2+4z^2$.

6.- Show that for all $\forall x,y,z\in\mathbb{R}$ it holds $x^2+y^2+z^2\geq xy+xz+yz$.

7.- Determine for what value of $\alpha\in\mathbb{R}$ the following quadratic forms are semidefinite indicating if it is positive or negative.

a) $Q(x,y,z)=x^2+2y^2+\alpha z^2-2xz$.

b) $Q(x,y,z)=x^2+\alpha y^2+\alpha z^2+2yz$.

8.- Classify, depending on the values of $\beta \in \mathbb{R}$, the quadratic form of expression $Q(x, y, z, t) = x^2 + y^2 + z^2 + t^2 + 2\beta yt + 2\beta xz$.

9.- Study, depending on the values of the real parameter a, the sign of the quadratic form of associated matrix $A = \begin{pmatrix} a & 2 & 0 \\ 2 & a & 3 \\ 0 & 3 & 3 \end{pmatrix}$.

10.- Consider the quadratic form of associated matrix $A = \begin{pmatrix} -1 & a & 2 & 1 \\ a & 0 & 1 & 0 \\ 2 & 1 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$. Is it negative definite for any value of the real parameter a?.

11.- Classify the following restricted quadratic forms:

a) $Q(x, y) = 2x^2 + y^2 + 2\sqrt{2}xy$ to $S = \{(x, y) \in \mathbb{R}^2 \mid x - \sqrt{2}y = 0\}$.

b) $Q(x, y, z) = x^2 + 4y^2 + 5z^2 + 2xy - 2xz + 4yz$ to the vectors (x, y, z) such that $x + 2y - z = 0$ and $2x - 3y + z = 0$.

c) $Q(x, y, z) = 2x^2 + y^2 - 4xy + 2yz$ to the vectors (x, y, z) such that $x - y + z = 0$.

d) $Q(x, y, z, t) = x^2 - z^2 + 2xz + xt + 2yz$ to the vectors (x, y, z, t) such that $x + y - z = 0,\ y - t = 0$.

12.- Classify the quadratic form $Q(x, y, z) = x^2 + y^2 - 2z^2$ restricted to:

a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + z = 0\}$.

b) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = -z\}$.

13.- Classify $Q(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$ restricted to:

a) $S = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2z = 0\}$.

b) $S = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

14.- Classify the quadratic form $Q(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ restricted to the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid x - y - 2z = 0\}$.

15.- Consider the quadratic form $Q(x, y, z) = 4x^2 + 5y^2 + z^2 - 4xz$.

a) Find a diagonal expression for Q.

b) Classify the sign of Q.

c) Classify Q restricted to $S = \{(x, y, z) \in \mathbb{R}^3 \mid x - z = 0\}$.

16.- Consider the quadratic form of associated matrix $A = \begin{pmatrix} 1 & 0 & b \\ 0 & 2 & 0 \\ b & 0 & 1 \end{pmatrix}$, where b is a real parameter.

a) Determine its sign as a function of the values of the parameter b.

b) Determine its sign restricted to the vectors (x, y, z) satisfying $y = 2z$, for any value of b.

17.- Let $Q(x, y, z) = y^2 - xy - xz - yz$.

a) Find a diagonal expression for Q.

b) Classify Q.

c) Classify, depending on the values of the real parameter α, Q restricted to the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = \alpha z\}$.

18.- Let $Q(x, y, z) = 2ax^2 + y^2 + z^2 + 4axz$, with $a \in \mathbb{R}$.

a) Classify Q depending on the values of the real parameter a.

b) If a = -1, find subsets S_1 and S_2 of $\mathbb{R}^3$ such that Q restricted to S_1 be positive definite and Q restricted to S_2 be negative definite.

19.- Considering the quadratic form Q (x, y, z) represented by the matrix $A = \begin{pmatrix} -2 & 3 & 0 \\ 3 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ answer reasonably the following questions:

a) Does the quadratic form Q admit the expression $Q(\tilde{x}, \tilde{y}, \tilde{z}) = -2\tilde{x}^2 - 2\tilde{y}^2 - 5\tilde{z}^2$?

b) Justify if there is any $(x_0, y_0, z_0) \in \mathbb{R}^3$ for which it is verified that $Q(x_0, y_0, z_0) > 0$.

c) Define, if possible, a subset $\mathbb{R}^3$ where the quadratic form Q is definite negative.

20.- Given the quadratic form $Q(x,y,z) = 2x^2 - y^2 - 4z^2 + 4zy$. It is requested:

a) Calculate a diagonal expression for Q.

b) Justify if there it exist some $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that $Q(x_0, y_0, z_0) < 0$.

c) Study the sign of Q constrained to $S = \left\{(x,y,z) \in \mathbb{R}^3 \mid y - 2z = 0\right\}$.

21.- Given the quadratic form $Q(x,y,z) = x^2 + 3y^2 + z^2 + 2xz$:

a) Classify Q (x, y, z).

b) Can $Q(\bar{x}, \bar{y}, \bar{z}) = \bar{x}^2 + 3\bar{y}^2 + 2\bar{z}^2$ be a diagonal expression of Q (x, y, z)?

22.- Given the high value of the public deficit of an imaginary country, the government decides to create a new tax *T* whose amount is a function of the payments (or reimbursements if applicable) of the income tax, *R*, and of the payments of the heritage tax, *P*, in such way that $T = 2R^2 + 4P^2 - 4RP$. The government, before setting up the new tax, wants to make sure that no taxpayer will be reimbursed by this new tax. Check that the new tax will fulfil its purpose with all and each one of the taxpayers.

23.- The economists of a company claim that the production function is of the type: $P = L^2 + K^2 - 2LK$, being *L* and *K* the number of workers and of machines, respectively. Besides that, it is known that each machine needs two workers for it to work. Check that indeed, *P* is a production function with the condition given.

24.- An investor estimates that by investing in three securities, A, B and C, the resulting portfolio will have a profitability of $U(x,y,z) = 2x^2 - 2y^2 - 7z^2 + 2xy + 6xz$, being x, y and z, the returns at the time of purchase of each value A, B and C respectively.

a) Analyze if said portfolio can generate losses.

b) Analyze how the previous response affects an economic situation in which the three products have the same profitability.

c) If the investor knows that the profitability of the product A is double of that of B, Will there be gains?

LINEAR PROGRAMMING

1.- Given the linear problems:

P1) $Min\,(Max) -2x_1 + 6x_2$

$s.t: \quad 2x_1 + 3x_2 \leq 12$

$-x_1 + 3x_2 \geq 6$

$x_1, x_2 \geq 0$

P2) $Min\,(Max) - x_1 + 2x_2$

$s.t: \quad x_1 - x_2 \leq 4$

$2x_1 - x_2 \geq -3$

$x_1 - 2x_2 \leq 4$

P3) $Min\,(Max) - x_1 + 2x_2$

$s.t: \quad x_1 \leq 4$

$x_1 + x_2 \leq 5$

$x_1 - 2x_2 \leq -2$

a) Represent the feasible set, the zero level curve of the objective function and the direction of growth of it at the point (0,0).

b) From the graphic representation, determine the vertices and analyze the existence of solution.

c) Solve each problem indicating the optimum values of the objective function and the point or points where they are attained.

2.- For the following problems:

a) Define the variables of the problem, the objective function to be optimized and the set of feasible solutions.

b) Draw a graphic representation of the feasible set.

c) Is the feasible set empty?, is it convex?, is it closed?, is it bounded?

d) How do influence the previous results in the existence and uniqueness of solution?

e) Solve graphically the problem, indicating the optimum value of the objective function.

E1) A certain company manufactures two types of products P1, P2. Each unit produced requires the utilization of two different types of machines M1, M2. The product P1 requires two hours of the machine M1 and an hour of the machine M2, whereas the product P2 requires M1 during an hour and 30 minutes and M2 during 15 minutes. Taking into account the available machines of both types, it is established that along the week there will be available 1200 hours of work in M1 and 375 of M2. Each unit of P1 leaves a profit of 600 m.u. and each unit of P2, 300 m.u. The owner of the company wishes to know the quantity of units that have to be produced of each product in order to maximize the weekly profit.

E2) A manufacturer produces chairs and tables for which it is required the utilization of two sections of production: the section of assembly and the section of painting. The production of a chair requires an hour of work in the section of assembly and two hours in the section of painting. On its side, the manufacture of a table needs three hours of assembly and an hour of painting. The section of assembly can be in operation only nine hours daily, whereas the section of painting only eight hours. The profit obtained producing a table is the double of that of producing a chair. What is the daily production of tables and chairs that maximizes the profit?

E3) A mining company produces lignite and anthracite. By the moment it is able to sell all the produced coal, the gain by ton of lignite and anthracite is of 4 and 3 m.u., respectively. The processing of each ton of lignite requires 3 hours of work of the machine to cut coal and other 4 hours of washing. On the other hand the processing of a ton of anthracite requires for the same tasks 4 and 2 hours respectively. The company has 12 and 8 hours available, respectively, for each one of these activities. Besides, it is supposed that at least 2 tons of coal should be produced daily. Pose a problem of linear programming with the aim of maximize the gain and solve it.

E4) A nutritionist advise an individual that suffers a deficiency of iron and vitamin B. The former indicates to the latter that he should ingest at least 2400 mgr. of iron, 2100 mgr. of vitamin B-1 and 1500 mgr. of vitamin B-2 during some period of time. There exist two pills of vitamins available, the make A and the make B. Each pill of the make A contains 40 mgr. of iron, 10 mgr. of vitamin B-1, 5 mgr. of vitamin B-2 and costs 6 cents. Each pill of the make B contains 10 mgr. of iron, 15 mgr. of vitamin B-1 and 15 mgr. of vitamin B-2 and costs 8 cents. Pose and solve a linear program that allows to compute the combination of pills that cover the nutritional requests at the lowest cost.

3.- Solve graphically the following linear programs and characterize the type of solution (unique, multiple–segment, multiple-half-line).

a) $\mathit{Max(Min)}\ 2x_1 + 3x_2$

$$\text{s.t:} \begin{cases} x_1 + x_2 \geq 1 \\ x_1 + x_2 \leq 3 \\ -x_1 + x_2 \leq 1 \\ -x_1 + x_2 \geq -1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

b) $\mathit{Max(Min)}\ 3x_1 + 2x_2$

$$\text{s.t:} \begin{cases} -x_1 + 2x_2 \leq 4 \\ 3x_1 + 2x_2 \leq 14 \\ x_1 - x_2 \leq 3 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

c) $\mathit{Max(Min)}\ 2x_1 + 3x_2$

$$\text{s.t:} \begin{cases} 7x_1 - x_2 \leq 5 \\ x_1 + 2x_2 \leq 4 \\ x_1 \geq 0 \end{cases}$$

d) $\mathit{Max(Min)}\ x_1 - \frac{1}{2} x_2$

$$\text{s.t:} \begin{cases} x_1 - x_2 \leq 4 \\ 2x_1 - x_2 \geq -3 \\ -x_1 + 2x_2 \geq -4 \end{cases}$$

4.- Solve using the simplex method the following linear programs.

a) $\mathit{Max}\ 2x_1 + 3x_2$

$$\text{s.t:} \begin{cases} x_1 + x_2 \geq 1 \\ x_1 + x_2 \leq 3 \\ -x_1 + x_2 \leq 1 \\ -x_1 + x_2 \geq -1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

b) $\mathit{Max}\ 3x_1 + 2x_2$

$$\text{s.t:} \begin{cases} -x_1 + 2x_2 \leq 4 \\ 3x_1 + 2x_2 \leq 14 \\ x_1 - x_2 \leq 3 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

c) $Max\ 3x_1 + x_2$

$$\text{s.t:} \begin{cases} x_1 + 2x_2 \le 5 \\ x_1 + x_2 - x_3 \le 2 \\ 7x_1 + 3x_2 - 5x_3 \le 20 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

d) $Min\ -5x_1 - 6x_2 + 6x_5$

$$\text{s.t:} \begin{cases} 5x_1 + 3x_2 + x_3 = 200 \\ 6x_1 + 5x_2 + x_4 = 300 \\ 3x_1 + 5x_2 + x_5 = 200 \\ x_i \ge 0 \quad i = 1, \ldots, 5 \end{cases}$$

5.- Check that the following linear programs do not have optimum solution, identifying if such a circumstance is due to not having feasible solutions or to the feasible set being not bounded in the direction of growth.

a) $Max\ 4x_1 + 3x_2$

$$\text{s.t:} \begin{cases} 3x_1 + 4x_2 \le 12 \\ 4x_1 + 2x_2 \le 8 \\ x_1 + x_2 \ge 4 \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

b) $Max\ x_1 + x_2 + x_3$

$$\text{s.t:} \begin{cases} -2x_1 + x_2 + x_3 = 1 \\ 2x_1 + x_2 \ge 2 \\ x_1 - 2x_2 \ge -2 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

6.- Write in standard form and the first table of the simplex algorithm for the following programs:

a) $Max\ 7x_1 + 9x_2 + 9x_3$

$$\text{s.t:} \begin{cases} x_1 \le 5 \\ x_2 \ge 1 \\ x_1 \ge x_2 + x_3 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

b) $Min\ x_1 + x_2 + x_3$

$$\text{s.t:} \begin{cases} -2x_1 + x_2 + x_3 = 1 \\ 2x_1 + x_2 \ge 2 \\ x_1 - 2x_2 \ge -2 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

c) $Max\ x_1 + 2x_2 - x_3$

$$\text{s.t:} \begin{cases} x_1 + 4x_2 + x_3 \le 5 \\ 2x_1 + x_2 - 3x_3 = 8 \\ x_1 \ge 0, x_2 \ge 0 \end{cases}$$

d) $Min\ 2x_1 - 3x_2 - x_3$

$$\text{s.t:} \begin{cases} x_1 - 4x_2 - 3x_3 \ge -3 \\ 2x_1 + x_2 + 4x_3 \ge 1 \\ x_1 + 2x_2 + x_3 = 8 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

e) $Min\ 12x_1 + 10x_2 + 10x_3$

$$\text{s.t:} \begin{cases} 2x_1 + x_2 + 3x_3 \ge 4 \\ 3x_1 + 4x_2 + x_3 \ge 2 \\ x_1 + 2x_2 + x_3 \ge 3 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

f) $Min\ -x_1 - 2x_2 + \frac{3}{2} x_3$

$$\text{s.t:} \begin{cases} x_2 - x_3 \le 1 \\ -x_1 + 4x_2 + 4x_3 \ge 12 \\ x_2 - 2x_3 \ge -1 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{cases}$$

7.- For the problem of linear programming:

$$Max \quad x_1 + x_2 + x_3$$

$$\text{s.t:} \quad \begin{cases} 2x_1 - 3x_2 + x_3 = 2 \\ x_1 - x_2 - 3x_3 = 5 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{cases}$$

Is the matrix formed by the first columns and second (x1, x2) a basic matrix of the problem? In the affirmative case, compute the basic feasible solution and the table associated to the mentioned solution. From the previous table, what can be said about the optimum solution of the problem?

8.- Complete the following table of a problem of minimization and interpret it for the different values of the parameter *t*:

		c_j	t	2	1	0
c_b	x_b	b	x_1	x_2	x_3	x_4
t	x_1	2	1	2	0	-1
1	x_3	3	0	2	1	-2
		z-c				

9.- The following problem of linear programming:

$$Min \quad -5x_1 - 2x_2 - 3x_3$$

$$\text{s.t:} \quad \begin{cases} x_1 + 5x_2 + 6x_3 \leq b_1 \\ x_1 - 5x_2 - 6x_3 \leq b_2 \\ x_i \geq 0, \ i = 1,2,3 \qquad b_j \geq 0, \ j = 1,2 \end{cases}$$

has as an optimum solution associated to the following table:

		c_j					
c_b	x_b	b	x_1	x_2	x_3	x_4	x_5
-5	x_1	30	1	p	6	1	0
0	x_5	10	0	q	-12	-1	1
		z-c	0	a	-27	d	E

a) Compute the components of the vector $b = (b_1, b_2)$ that provide the mentioned solution, knowing that the initial basic feasible solution was formed by the variables x_4 and x_5.

b) Complete the table.

10.- Let the program

$$\begin{aligned} &Max \quad 2x_1 + 3x_2 + 2x_3 \\ &\text{s.t:} \quad \begin{cases} 2x_1 + 2x_2 + x_3 + x_4 = 6 \\ x_2 + x_3 + x_5 = 4 \\ x_i \geq 0, \ i = 1,\ldots,5 \end{cases} \end{aligned}$$

Knowing that the following table is the one corresponding to the basic feasible solution (1, 0, 4, 0, 0), complete the former and interpret it.

		c_j					
c_b	x_b	b	x_1	x_2	x_3	x_4	x_5
2	x_1		1	$1/2$	0	$1/2$	
2	x_3	4	0	1	1		1
			2		2	1	1
		z-c	0		0	1	1

11.- Considering that $b_2 \geq 0$ in the problem:

$$\begin{aligned} &Max \quad c_1x_1 + 2x_2 + x_3 \\ &\text{s.t:} \quad \begin{cases} x_1 + 2x_2 + a_{13}\, x_3 \leq 8 \\ 2x_1 + 4x_3 \leq b_2 \\ x_1, x_2, x_3 \geq 0 \end{cases} \end{aligned}$$

and knowing that the following values of the optimum table (x_4, x_5 are slack variables)

		c_j					
c_b	x_b	b	x_1	x_2	x_3	x_4	x_5
	x_2					1/2	
	x_1	3					
		z-c			7	1	1

without applying the simplex method, compute c_1, a_{13}, b_2 and complete the table.

12.- A farmer possesses a small holding of 640 m^2 to devote it to the crop of three types of fruit trees: orange trees, pear trees and apple trees. He wonders about the way of divide up the surface of the small holding amongst the three varieties to achieve the maximum profit knowing that each orange tree needs 16 m^2, each pear tree 4 m^2 and each apple tree 8 m^2; the farmer can work a maximum of 900 hours of work/year in total, meanwhile each orange tree needs 30 hours/year, each pear tree 5 hours/year and each apple tree 10 hours/year. The unitary profits of each orange tree, pear tree and apple tree are 50, 25 and 20 m.u. respectively.

13.- A veterinary advises a rancher who breeds chickens, a minimum diet for the feeding of the birds composed of 3 units of iron and 4 units of vitamins. The rancher knows that each kilo of corn provides 2.5 units of iron and 1 unit of vitamins, each kilo of fish flour 3 units of iron and 3 units of vitamins, and each kilo of synthetic feed 1 unit of iron and 2 units of vitamins. The rancher wonders about the composition of the optimum diet that minimize the costs of the feeding, knowing that the prices of the corn, fish flour and synthetic feed are, respectively, of 20, 30 and 16 monetary units.

14.- A company elaborates two products, A and B, that it sells by 3 and 4 monetary units (each unit), respectively. In order to choose the produced quantity it has to be taken into account the following conditions: 1) The market demands a minimum quantity of 300 units of each product; 2) The quantity produced of A has to be at most the triple of the one of B; 3) the maximum production of the product B is 600 units. Formulate and solve the problem of linear programming that allows to determine the quantities that have to be produced in order to maximize the income, taking into account the previously established conditions.

15.- A transport company has 720 m.u. for the acquisition of new vehicles. After a study, it has chosen two types of vehicles for the purchase. The vehicle A has a capacity of 12 Tm and achieves a speed of 65 km/h, being its cost 12 m.u., whereas vehicle B has a capacity of 22 Tm, with a speed of 50 km/h and a cost of 18 m.u. The vehicle A require only a driver in each turn of work and supposing that the vehicle A does three turns per day and can circulate 18 h/day; whereas the vehicle B requires a team of two men circulating 21 h/day with three turns of work per day. The company currently has 210 drivers only and does not want to increase this number. Formulate a problem of linear programming with the aim to find the number of vehicles of each type that have to be bought if the company wishes to maximize its capacity of tonnage per kilometer and day.

16.- A company can produce three different articles using two raw materials, A and B. To manufacture a unit of the first article it needs 200 units of A and 300 units of B; for the second it needs 150 of A and 240 of B; and for the third it needs 100 of A and 150 of B. The raw materials' supplier establishes as a condition a minimum demand for A of 5600 units and for B of 8700 units. Knowing that the unitary cost to produce the first article is 90 monetary units (m.u.), the one for the second 80 m.u. and the one for the third 50 m.u., the idea is to compute the quantities of these three articles to be produced with the minimum cost.

a) Pose the corresponding problem of linear programming.

b) Present the first table of the simplex of the mentioned problem.

c) Pose the dual problem and solve it by the simplex algorithm.

d) On the basis of the solution of the dual, obtain the solution of the primal.

e) Give the economic interpretation of the solution both of the primal and the dual.

17.- A building material company produces, amongst other things, three types of cement that need for their elaboration to be processed through two departments: preparing and mixing. The technical requests of a ton of cement in each department as well as its net profits are given in the following table:

Type of cement	Hours of Preparing	Hours of Mixing	Net profit
C1	6	3	3
C2	3	4	1
C3	5	5	5

The department of preparing can work 45 hours per week maximum, and the department of mixing 30 hours per week. (It is supposed that the number of tons to be manufactured does not have to be necessarily an integer number).

a) Pose the problem of linear programming that yields the optimum production of cement, that is to say, the number of tons of each type of cement that maximize the profit.

b) Compute the maximum profit, indicating the combination of cements that produce the mentioned maximum profit.

c) The department of marketing wishes to know if it would result worthwhile the production of a new type of cement whose profit by ton is 6 m.u. and that needs 6 hours of preparing and 5 hours of mixing per each ton produced. Will be it worthwhile to manufacture this new cement?, if yes, in what quantity?

d) Answer to the questions posed in c) when the profit in monetary units by ton of the new cement is 5, taking into account besides that if the new cement is produced, the company's public image improves.

18.- A rancher possesses 100 hectares to cultivate wheat and canary grass. The cost of the seed of wheat is 4 euros per hectare and the seed of canary grass has a cost of 6 euros per hectare. The total cost of the workforce is 20 and 10 euros per hectare respectively. The expected profit is 110 euros per hectare of wheat and 150 euros per hectare of canary grass. If the rancher does not wish to spend more than 480 euros in seeds neither more than 1500 euros in workforce:

a) Pose the linear program that allows to maximize the profit.

b) How many hectares are to be devoted to each one of the crops in order to obtain the maximum profit? (Use the simplex algorithm to give the answer).

c) The rancher wonders about whether it is interesting to cultivate a new cereal that has a cost of 5 euros per hectare and a workforce cost of 15 euros per hectare, expecting a profit of 130 euros per hectare. Reason out the answer.

d) Determine the maximum profit and the crops plan if the new cereal gives a profit of 135 euros per hectare.

e) Determine the maximum profit and the crops plan if the farmer invests 500 euros in seeds.

19.- A company produces and commercializes two articles A and B that provide the former unitary profits of 30 and 50 m.u., respectively. The articles A and B are obtained by means of the transformation of the minerals P and Q, in such a way that each unit of A requires 2 units of P and 3 of Q. Each unit of B requires 4 units of P and 3 of Q. Each day, 60 units of P and 80 of Q are available. Supposing that the company sells all of the production:

a) Pose the linear program of profits' maximization, its dual problem and write the first table of the simplex of the primal problem.

b) Knowing that (using the simplex algorithm for maximum) the last table is:

		c_j				
c_b	x_b	b	x_1	x_2	x_3	x_4
50	X_2	$10/3$	0	1	$1/2$	$-1/3$
30	X_1	$70/3$	1	0	$-1/2$	$2/3$
		$2600/3$	30	50	10	$10/3$
		z-c	0	0	10	$10/3$

Compute the interval of availability of the mineral Q for which the previous table continues to be optimum.

c) How much would pay the company by 3 additional units of mineral Q?. And by 30?

d) Give the solution of the dual problem.

e) It would be interesting to manufacture a new product C that consumes 3 units of mineral P and 4 of mineral Q, with a unitary profit of 45 m.u? In the affirmative case, determine the optimum production plan.

20.- A toy company manufactures two types of wood toys: trucks and trains. A truck is sold by 27 monetary units (m.u.) and it uses raw material worth 10 m.u. Besides, the company has evaluated in 14 m.u. the sum of indirect costs and workforce costs of the manufacture of each truck. For each train the previous quantities are 21, 9 and 10 m.u., respectively. The manufacture of the trucks and trains needs two types of work: finishing and carpentry. A truck needs two hours of finishing and 1 hour of carpentry. A train needs an hour of finishing and another one of carpentry. Each week, the company has all the raw material it needs, but only will have 100 hours of finishing and 80 of carpentry. The demand of trains is boundless, but the company will only be able to sell a maximum of 40 trucks per week. Formulate and solve a linear mathematical program that reflects this situation and allows the company to schedule the weekly production in order to maximize the profit (income less expenses).

21.- A shop sells markers by 20 cents of euro and notebooks by 30 cents of euro. We have 150 cents of euro and intend to buy, at least, the same notebooks than markers.

a) Which will be the maximum number of products that we can buy?

b) Determine the interval of variation of the cost of the notebooks so that the final table in the simplex algorithm of a) remains optimum.

c) Determine the interval of variation of the coefficient "number of notebooks to buy" so that the final table of a) remains optimum.

d) Suppose that the constraint "intend to buy, at least, the same notebooks that markers" change to "the difference between the number of markers and notebooks is lower than or equal to three units"; in this case, which is the new solution?

22.- The director of a company observes that his only sales agent never surpasses the 5 million monthly in sales and thus the former has decided to hire two new sales agents that would help the latter. After a test period the director arrives to the conclusion that he will have to pay to each new vendor a commission of 10% of the sales that they achieve. Additionally, the director will require a minimum monthly sales of 1 million of one of the new sales agents, who in turn, will achieve an extra commission of 5% of the sales. Besides, by previous experience the director knows that the initial sales agent will sell equal or more than the two new sales agents together. Knowing that the initial sales agent earns a 30% commission on his own sales, the director of the company wishes to know the monthly maximum profit that can be obtained with the three sales agents if the explained circumstanced hold. In such a case, indicate the monthly distribution of the sales between the three sales agents.

23.- A youngster has 74 € for leisure expenses each 4 weeks, and spend that quantity completely. All the expenses are distributed in beers, in going to the cinema or in going to have dinner. The cost of a beer is €1, the one of each cinema ticket 6 € and every time that he goes to have dinner he spends 10 €. It is the habit of the youngster to go at least once to the cinema per week and never goes to have dinner more than once in the 4 weeks. The satisfaction that he obtains for going to the cinema is equal to the one of having 5 beers and the one of going to have dinner is equivalent to having 15 beers. Warning: take into account that the available budget is to be spent completely.

a) Obtain the problem of linear programming that allows to maximize the satisfaction and solve it by the simplex method.

b) Find the dual problem of the initially posed problem, and the solution of the dual.

c) Compute the interval of the satisfaction obtained with each beer so that the youngster do not change his habits.

d) How much has to vary his budget so that he stops spending in the three leisure alternatives?

24.- A company produces and commercializes two products P1 and P2 which are obtained by means of the use of two workshops A and B that manufacture indistinctly the components of both products.

The available monthly capacity of the workshop A is such that, in case to devote it exclusively to the manufacture of components of P1, would obtain those enough to manufacture 50 units of this product. In the opposite case (exclusive dedication to the components of P2) could obtain 100 units of P2. On the other hand, the monthly capacity of the workshop B is sufficient to provide, at most, the number of necessary components for obtaining a total of 60 units of product finished, independently of the mixing of the products P1 and P2.

The unitary sale prices of P1 and P2 are, respectively, 42 and 35 m.u., and the unitary costs of them are 30 and 25 m.u., respectively.

According to these premises the company manager wishes to know:

a) Which is the program of production that would drive to an optimum result for the monthly management? (Maximum profit).

b) Under what conditions the company would be interested in devoting part of his workshops for a distinct activity than for the products P1 and P2?

25.- A company is devoted to the elaboration of two products P1 and P2 that provide profits of 50 m.u./m^3 and 60 m.u./m^3, respectively. The mentioned elaboration yields two types of toxic gases G1 and G2 that are evacuated to the atmosphere in the proportion indicated in the following table:

	Emission of toxic gases in liters by m^3	
Type of toxic gas	P_1	P_2
G_1	24	36
G_2	8	12

Because of new pollution legislation, the daily emission of G1 and G2 will not have to surpass the 600 and 800 liters, respectively.

The director of production of the company can adopt an anti-pollution mechanism in the process of manufacture of P1 and/or P2, and can reduce the toxic gases G1 and G2 in a 75% and in a 50% respectively, independently of the process the mechanism is applied to.

The company manager wishes to know which would be the program yielding the quantities of P1 and P2 that have to be obtained daily with the aim to maximize the profit without breaking the legislation. The use of the cited mechanism in any of the manufacturing processes produces a decrease of 10 m.u./m^3 in the obtained profit of the corresponding product.

26.- A company manufactures three types of products X, Y and Z whose productive processes involve the use of three sections. In the following tables they are shown the times employed together with the monthly availabilities of the different sections and the estimated cost of the idle time for each one of them:

	Time used by each unit of product		
Section	X	Y	Z
S_1	2h	2h	2h
S_2	2h	-	6h
S_3	4h	1h	-

Section	Monthly maximum availability in S_i	Cost of the idle hour in S_i
S_1	228	12 m.u.
S_2	198	8 m.u.
S_3	279	4 m.u.

Knowing that the unitary profits of X, Y and Z are respectively 8, 9 and 13 m.u. we wish to know the linear program that would yield the maximum monthly profit.

27.- Pose the dual of the following problem, and if it is possible, compute graphically its solution.

$$\begin{aligned}&\text{Max} \quad x_1 + x_2 + x_4\\ &\text{s.t:} \quad \begin{cases} 2x_1 + x_2 + 4\,x_3 = 10 \\ x_1 + 2x_4 \le 8 \\ x_1, x_2 \ge 0; \quad x_3 \le 0; \quad x_4 \ \ \textit{unconstrained} \end{cases}\end{aligned}$$

28.- It is known that the optimum solution of the linear problem

$$\begin{aligned}&\text{Max} \quad x_1 + 3x_2\\ &\text{s.t:} \quad \begin{cases} x_1 + 4\,x_2 \le 100 \\ x_1 + 2x_2 \le 60 \\ x_1 + x_2 \le 50 \\ x_1, x_2 \ge 0 \end{cases}\end{aligned}$$

is formed by the variables x_1, x_2, x_5, (slack variable of the third restriction), and that B^{-1}=

$$A_b^{-1} = \begin{pmatrix} -1 & 2 & 0 \\ 1/2 & -1/2 & 0 \\ 1/2 & -3/2 & 1 \end{pmatrix}.$$

a) Pose the dual problem of the given problem and the solution of the dual.

b) Perform a sensitivity analysis for b_1 (first component of the right side of the constraints).

c) Perform a sensitivity analysis for c_2 (coefficient in the objective function associated to the variable x_2).

d) It is posed the possibility of manufacturing a new product x_n so that the resultant problem is written:

$$\begin{aligned}&\textit{Max} \quad x_1 + 3x_2 + 4x_n\\ &\text{s.t:} \quad \begin{cases} x_1 + 4\,x_2 + 4x_n \le 100 \\ x_1 + 2x_2 + 3x_n \le 60 \\ x_1 + x_2 + 3x_n \le 50 \\ x_1, x_2, x_n \ge 0 \end{cases}\end{aligned}$$

Without solving “the new problem” by the simplex algorithm compute the optimum solution and the maximum value of the objective function.